Entanglement and gravitational physics

2006 J. Phys. A: Math. Gen. 39 6385

(http://iopscience.iop.org/0305-4470/39/21/S32)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 6385–6391 doi:10.1088/0305-4470/39/21/S32

Entanglement and gravitational physics

Dmitri V Fursaev

Dubna International University and Bogoliubov Laboratory of Theoretical Physics,
Joint Institute for Nuclear Research, 141 980 Dubna, Moscow Region, Russia

E-mail: fursaev@thsun1.jinr.ru

Received 4 November 2005, in final form 8 December 2005
Published 10 May 2006
Online at stacks.iop.org/JPhysA/39/6385

Abstract
It is argued that the entanglement entropy in condensed matter systems can be used to study different aspects of quantum gravity, such as universality of the low-energy physics, the renormalization group behaviour of the gravitational coupling and the statistical meaning of the Bekenstein–Hawking entropy.

PACS numbers: 04.60.−m, 03.70.+k, 03.65.Ud, 05.50.+q

1. Introduction

As was first suggested in [1], condensed matter systems enable one to model quantum effects in external gravitational fields. Under certain conditions, sound waves in liquid helium or in Bose–Einstein condensates behave as scalar excitations, propagating in an effective curved background with a metric similar to that near a black hole horizon (see, e.g., [2] and references therein).

In this paper, we argue that condensed matter systems can also be used to study other aspects of quantum gravity. Our suggestion is based on the properties of entanglement entropy. Entanglement entropy is introduced as the measure of information loss about quantum states which cannot be observed. The states can be located in a region of space separated from the observable states by a boundary $\mathcal{B}$.

Consider, for instance, a lattice of spins being in a quantum state characterized by a density matrix $\hat{\rho}$. Suppose that the lattice is divided into regions A and B with a common boundary $\mathcal{B}$. The entanglement between the two regions can be described by the reduced density matrix $\hat{\rho}_B = \mathrm{Tr}_A\,\hat{\rho}$, where the trace is taken over the states of spin operators at the lattice sites in one of the regions (for instance, the region A). The entanglement entropy is defined as the von Neumann entropy in the region B as $S = -\mathrm{Tr}_B\,\hat{\rho}_B \ln \hat{\rho}_B$. Standard arguments show that S does not depend on whether the reduced matrix is obtained by tracing over the states in A or B.

A simple but non-trivial example is the entanglement entropy for a block of spins in the Ising model. The Hamiltonian of the model is

$$H = \sum_{i=1}^{N} \left(\sigma_i^x \sigma_{i+1}^x + \lambda \sigma_i^z\right), \tag{1}$$

where N is the number of spins and σ_i^x, σ_i^z are the Pauli matrices. The parameter λ is the strength of the external magnetic field. At zero temperature, the Ising chain has a second-order phase transition at the critical value $\lambda = 1$.

The entanglement entropy can be investigated for the ground state when the chain is separated into two blocks of contiguous spins of equal sizes $N/2$. Suppose that λ is fixed and N varies (N is much larger than the correlation length). The results show that there are two regimes [3]. In the off-critical regime, $\lambda \neq 1$ and $|\lambda - 1| \ll 1$, the entropy at large N reaches the saturation value $S(N, \lambda) = -\frac{1}{6}\log_2 |\lambda - 1|$. In the critical regime, $\lambda = 1$, the entropy does not reach saturation and behaves at large N as $S(N, \lambda) \simeq \frac{1}{6}\log_2 N/2$.

The explanation of these results (as will be shown in the next section) is based on the fact that in the continuous limit, $N \to \infty$, near the critical point the Ising model corresponds to a quantum field theory (QFT) with a fermion field whose mass (the inverse correlation length) is monotonically related to $|\lambda - 1|$. In the critical regime, the mass vanishes and one has a fermionic conformal field theory (CFT) with the total central charge $c = 1$.

Higher dimensional condensed matter systems have similar properties. There are a number of systems which are described near a critical point by relativistic QFTs with massive fields. For those systems the entanglement entropy (in spaces having number of dimensions larger than 1) is a geometrical quantity. If $\mathcal{B}$ is a simple plane or a sphere, the entropy is proportional to the area of $\mathcal{B}$ [4]. This property follows from the fact that field excitations in A and B are correlated only on the boundary.

Our arguments relating the entanglement entropy and gravitational physics are based on the observation that higher dimensional quantum many-body systems near a critical point set an example of 'induced gravity theories'. As in Sakharov's approach [5], the corresponding Einstein action is entirely induced by quantum effects of field degrees of freedom. Let us emphasize that, because we are interested only in the behaviour of the Newton coupling, we do not need to introduce any metric (or its analogue) in the condensed matter system. To determine the coupling, it is sufficient to study the response of the effective action in a flat space to the introduction of a conical singularity. This is equivalent to the definition of the gravitational coupling in these theories as the entanglement entropy per unit area. Hence, the coupling is a computable quantity which can be studied by analytical and numerical methods; and from this analysis one can get new insights into quantum gravity phenomena.

2. Geometrical derivation of the entanglement entropy

For further discussion, it is instructive to recall how the results for the 1D Ising chain can be obtained from geometrical properties of $S(N, \lambda)$. The definition of the entropy can be rewritten by using the formula

$$S = -\lim_{n\to 1} \frac{\partial}{\partial n} \mathrm{Tr}_B\, \hat{\rho}_B^n. \tag{2}$$

Near the critical point the system is described by a field theory with field variables ϕ. The density matrix $\hat{\rho}_B$ in the configuration representation depends on variables ϕ_B on the interval B. Suppose that the system is in a state with the temperature T. (The result for the ground state

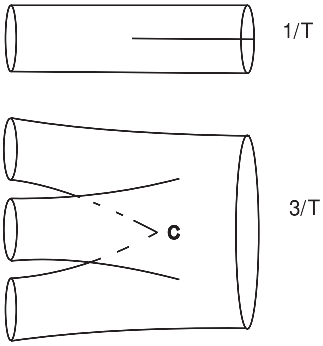

Figure 1. The upper picture shows the cylinder with the circumference length T^{-1} and a cut along the axis. The space $\mathcal{M}_\beta$ is schematically drawn for $n = 3$ on the lower picture. It is obtained by gluing along the cuts of three copies of the cylinder. The circumference length of the right boundary of $\mathcal{M}_\beta$ is $3T^{-1}$. The cuts meet at the point C which is a conical singularity.

entanglement can be recovered in the limit $T \to 0$.) Then the matrix elements of $\hat{\rho}_B$ can be described in terms of the Euclidean path integral

$$\langle \phi'_B | \hat{\rho}_B | \phi_B \rangle = \mathcal{N}^{-1} \int [D\tilde{\phi}]\, \mathrm{e}^{-I_E[\phi_B, \phi'_B]}. \tag{3}$$

$\mathcal{N}$ is a normalization coefficient introduced to satisfy the condition $\mathrm{Tr}_B\, \hat{\rho}_B = 1$. The classical action $I_E(\phi_B, \phi'_B)$ is defined on a Euclidean space with a compact time. In two dimensions, this space is a cylinder with the circumference length T^{-1} and a cut parallel to its axis; the length of the cut is equal to the length of the interval B (see the upper picture in figure 1). The integration in (3) implies that $\tilde{\phi}$ take values ϕ_B and ϕ'_B on the different sides of the cut. If n is positive and integer, the matrix elements of the operator $\hat{\rho}^n_B$ can be represented by integral (3) where field variables are defined on an n-sheeted surface with the same cut. The cut disappears when one takes the trace.

Thus, the field variables in the path integral representation of quantity $\mathrm{Tr}_B\, \hat{\rho}^n_B$ are defined on a manifold $\mathcal{M}_\beta$, where $\beta \equiv 2\pi n$. This manifold is locally flat but has a non-trivial topology. The important property is that $\mathcal{M}_\beta$ has a singularity at the point (or generally at a hyper-surface) where all n cuts meet. In two dimensions this is a conical singularity because a unit circle around it has a circumference length larger than 2π.

One can rewrite (2) in another form following from (3):

$$S = \lim_{\beta \to 2\pi} \left(\beta \frac{\partial}{\partial \beta} - 1 \right) \Gamma(\beta), \tag{4}$$

where $\Gamma(\beta)$ is the effective action defined as

$$\mathrm{e}^{-\Gamma(\beta)} = \int [D\tilde{\phi}]\, \mathrm{e}^{-I_E(\beta)} \tag{5}$$

$(\Gamma(2\pi) = -\ln \mathcal{N})$. Functional $I_E(\beta)$ is the classical action on $\mathcal{M}_\beta$. After computing $\Gamma(\beta)$ one can replace the discrete parameter β with a continuous one and use (4). This can be done even if $\mathcal{M}_\beta$ itself cannot be defined at an arbitrary β.

The expression of the entanglement entropy in terms of the effective action has a number of advantages. It enables one to reformulate the problem in geometrical language and use powerful methods of a spectral geometry to study the structure of the ultraviolet divergences of $\Gamma(\beta)$ and S. The ultraviolet divergent part, $\Gamma_{\mathrm{div}}(\beta)$, depends non-trivially on β. For instance, in the dimensional regularization in two dimensions $\Gamma_{\rm div}(\beta)$ contains the term $a_1(\beta)/(4\pi\varepsilon)$, where ε is a regularization parameter and

$$a_1(\beta) = \frac{\pi}{3}\left(\frac{2\pi}{\beta} - \frac{\beta}{2\pi}\right) + \frac{1}{3}\int_{\partial\mathcal{M}_\beta} k. \tag{6}$$

Here k is the extrinsic curvature of the boundary $\partial\mathcal{M}_\beta$ of $\mathcal{M}_\beta$. Equation (6) holds for free scalar or spinor fields.

The non-trivial dependence on β in the first term in the rhs of (6) is a direct consequence of the conical singularity. In the spectral geometry theory, $a_1(\beta)$ is one of the coefficients of the asymptotic expansion of the trace of the corresponding heat-kernel operator [6].

For $\beta = 2\pi n$ in two dimensions, $\mathcal{M}_\beta$ looks like 'pants' with n legs (see the lower picture in figure 1). Its boundary consists of n circles with length T^{-1} each and a circle of length nT^{-1}. However, because the extrinsic curvature k for each boundary is zero, the coefficient $a_1(\beta)$ is determined only by the conical singularity.

In the critical regime when the theory is a CFT the dependence on the size of the cylinder L (provided the size of the subsystem is $L/2$ and T goes to zero) is entirely determined from the anomalous scaling: the renormalized actions for systems of sizes L and L_0 differ by the term $-\frac{a_1(\beta)}{4\pi}\ln L/L_0$. It is this term which results in the scaling law of the entanglement entropy in the critical Ising model. This can be easily seen with the help of (4) if one puts $L \sim N$.

In the off-critical regime, $m \gg T, m \gg L^{-1}$, the effective action is approximated by the following leading terms:

$$\Gamma(\beta) \simeq -\frac{\sigma\beta}{16\pi^2}\frac{L\Lambda^2}{T} + \frac{a_1(\beta)}{8\pi}\ln\frac{m^2}{\Lambda^2}, \tag{7}$$

where Λ is the ultraviolet cutoff, $\sigma = +1$ for scalars and -2 for spinor fields. The first term in the rhs of (7) is a 'vacuum energy' proportional to the volume of $\mathcal{M}_\beta$. Substitution of (7) into (4) yields $S \simeq -\frac{1}{6}\ln m \simeq -\frac{1}{6}\ln|\lambda - 1|$, in accord with numerical results. The non-zero value of S is again ensured by the conical singularity.

3. Relation to gravity

The reason why the entanglement entropy is related to the gravity theory is because the conical space possesses a curvature concentrated on the tip. For the tip of the cone at $x = 0$, the curvature is the distribution $R = 2(2\pi - \beta)\delta^{(2)}(x)$, see, e.g., [7]. By taking into account this fact and (6), one can represent the effective action (7) in the limit β close to 2π as

$$\Gamma(\beta) = \frac{1}{4G^{(2)}(m)}\left(\int_{\mathcal{M}_\beta}(R + 2\lambda^{(2)}) + \int_{\partial\mathcal{M}_\beta} 2k\right). \tag{8}$$

This is a two-dimensional analogue of the Einstein–Hilbert action with some induced cosmological, $\lambda^{(2)}$, and Newton, $G^{(2)}$, couplings. We are interested here in the Newton coupling which is defined as

$$\frac{1}{G^{(2)}(m)} = \frac{1}{12\pi}\ln\frac{\Lambda^2}{m^2} = \frac{1}{\pi}S. \tag{9}$$

Therefore, by studying the entanglement entropy in a condensed matter system in the near-critical regime one studies the properties of the induced gravitational coupling in an effective gravity theory. The coupling is determined by the response of the effective action to the conical singularity.

Certainly, (8) cannot be considered the true gravity theory: the curvature term in the action in combination with the boundary term yields a topological invariant. The situation changes in higher dimensions. Consider, for instance, a 3D quantum theory on a cubic lattice (with the edge size L) which in the continuum limit is described by a free-scalar QFT in the (3+1)-dimensional spacetime. The result for the entropy in this theory can be obtained from formulae derived in two dimensions [8]. Suppose that the entanglement entropy is calculated when the lattice is divided into two regions with the quadrangle boundary $\mathcal{B}$ of the size L by L. The field variables $\phi(x, y)$ depend on two types of coordinates. Coordinates y^i $(-L/2 \leqslant y^i \leqslant L/2$, $i = 1, 2)$ correspond to directions along the boundary, while x^α, $\alpha = 0, 1$, are the time and a direction orthogonal to $\mathcal{B}$. A four-dimensional field is equivalent to an infinite tower of 2D fields $\phi_{\mathbf{p}}(x)$ defined by a Fourier transform in y coordinates. The index $\mathbf{p}$ is a momentum of the field along $\mathcal{B}$. If $\phi(x, y)$ has the mass m the 2-field $\phi_{\mathbf{p}}(x)$ has the mass $m(p) = \sqrt{p^2 + m^2}$, $p = |\mathbf{p}|$. The effective action of the theory $\Gamma(\beta)$ is the sum of the actions of all p-modes. Each 2D action at β close to 2π is given by (8) with coupling $G^{(2)}(p)$ defined in formula (9) where m is replaced by $m(p)$. It is not difficult to see that $\Gamma(\beta)$ takes the correct form of the Einstein–Hilbert action. The action contains the integral of the four-dimensional curvature which is $\int^{(4)} R = 2(2\pi - \beta)\mathcal{A}$, where $\mathcal{A} = L^2$ is the area of $\mathcal{B}$. The coefficient by this integral yields the induced Newton constant G:

$$\frac{1}{G} = \int_0^\Lambda \frac{2p\,\mathrm{d}p}{G^{(2)}(m(p))} = \frac{\Lambda^2}{12\pi} + \frac{m^2}{12\pi} \ln \frac{m^2}{\Lambda^2}. \tag{10}$$

By applying (4), one finds the entanglement entropy $S = \mathcal{A}/(4G)$. We rewrite this relation as

$$G^{-1} \equiv \frac{4S}{\mathcal{A}} \tag{11}$$

and use it as a definition of the Newton coupling in the induced gravity theory corresponding to a given condensed matter system. The definition is applicable to the case when $\mathcal{B}$ is a simple plane (which we assume). Equation (11) enables one to study a number of problems, some of which will be discussed in section 4.

There are different higher dimensional many-body systems which can be used as analogue models of the induced gravity theories. An important subclass among them is higher dimensional Ising models which are interesting for several reasons. First, in the critical regime, Ising models are equivalent to scalar field theories with self-interactions. Second, the behaviour of most known second-order phase transitions is equivalent at the critical point to a three-dimensional (3D) Ising model. Third, there are indications that Ising models can be represented as theories of random (hyper) surfaces. In particular, as was conjectured in [9] near the point of the second-order phase transition the 3D Ising model might be equivalent to a non-critical fermionic string theory.

The two-dimensional Ising model is exactly solvable. The reduced density matrix for the ground state entanglement can be diagonalized, which significantly simplifies the computations. In higher dimensions exactly solvable Ising models are not known. Thus, one has to develop numerical methods for computing the entanglement entropy.

4. Applications

4.1. RG flow of the gravitational coupling

The knowledge of the entanglement entropy as a function of parameters of the theory can be used to find the renormalization group (RG) evolution of the induced Newton constant (11). This information is important for understanding the behaviour of gravitational interactions at different scales.

Consider as an illustration a one-dimensional spin chain. It has a compact momentum space with the radius $\bar{p} = 2\pi/a$ determined by the lattice spacing a. The Wilson RG transformation implies integration over high-energy modes with momenta in the interval $t^{-1}\bar{p} \leqslant p \leqslant \bar{p}$ (with $t > 1$) followed by rescaling, p to $p' = tp$. As a result, one gets a theory with larger masses. For the Ising model (1) with $\lambda \neq 1$ this is equivalent to increasing the difference $|\lambda - 1|$. The RG transformation drives the theory away from the ultraviolet fixed point $\lambda = 1$.

The RG evolution of the entanglement entropy for the Ising model is known precisely [10]. It is in accord with the general property that the entanglement entropy in unitary theories should not increase along the RG flow because RG transformations eliminate the contribution of the high-energy modes.

The fact that the entropy is not increasing does not imply the same property for the induced coupling G^{-1} defined by (11). The coupling behaves as the density of the entropy, $G^{-1}(t) = t^{D-2}\mathcal{S}(t)$, where D is the number of spacetime dimensions and $\mathcal{S}(t)$ is a function which has the same RG evolution as the entropy. Thus, non-decreasing of S does not contradict the fact that gravitational interactions get weaker in the infrared region.

4.2. The scaling hypothesis

The scaling hypothesis in classical critical phenomena asserts that the physics is determined by large-scale fluctuations which do not depend on the underlying microscopical details. For instance, in a magnet at temperature T in an external field h which undergoes a second-order phase transition, the physics is explained by the large domains of aligned spins. The microscopic atomic scale does not enter into thermodynamical relations near the critical point $T = T_c$. Non-analytic dependence of the physical quantities on $|T - T_c|$ is entirely determined by a 'singular part' of free-energy density f_{sing}. This part has a universal scaling $f_{sing}(t, h) = \xi^{-(D-1)} f_{\pm}(h\xi^{-d_h})$, where $\xi \sim |T - T_c|^{-\nu}$ is the correlation length and d_h is the scaling dimension of h. Functions $f_{\pm}$ are universal in the sense that they coincide with different systems from the same class.

It was conjectured in [11] that near a critical point, in analogy with the classical scaling, the entanglement entropy per unit area has a 'singular part' $s_{sing}(g, h, T) = \xi^{-(D-2)} s_{\pm}(h\xi^{-d_h}, T\xi^{-z})$, where $\xi = |g - g_c|^{-\nu}$ is the correlation length, g is a parameter driving the phase transition at g_c, h is an external field with scaling dimension d_h and T is the temperature. The universality conjecture means that $s_{\pm}$ are universal functions. Note that s_{sing} is not the total entropy density. There exists a piece in the entropy depending on the lattice spacing a, which as was suggested in [11] is analytic in g.

Clearly, in the context of the induced gravity, the universality hypothesis is a statement about the gravitational coupling.

4.3. The problem of the black hole entropy

Studying entanglement in quantum critical phenomena might be helpful for understanding the microscopical origin of the Bekenstein–Hawking entropy of black holes. If the gravity is entirely induced by some underlying degrees of freedom, the entropy of a black hole can be related to the entanglement between observable states and states hidden inside the horizon (see, e.g., [8, 12] and references therein). Understanding the relation between the two entropies in the framework of a local relativistic quantum field theory is plagued by the problem of ultraviolet divergences. The definition of the induced Newton coupling requires either introduction of the ultraviolet cutoff or working with a special class of ultraviolet finite theories with non-minimal couplings to the curvature in the Lagrangians. The presence of non-minimal couplings makes statistical interpretation of the Bekenstein–Hawking entropy a more difficult task. In the suggested condensed matter analogues such problems do not exist. The fundamental microscopical theory is not a field theory. It has a natural cutoff, a lattice spacing, and the corresponding gravitational coupling is a microscopically computable quantity. Also the non-minimal couplings do not occur at least for models described in the critical regime by fermionic theories.

For the given analogue models, the entanglement entropy associated with the black hole horizon is identical to the Bekenstein–Hawking entropy. Therefore, one has a chance to learn what the real degrees of freedom of a black hole are. Are they spin variables, non-critical stings (as for the 3D Ising model) or something else? One can also ask other questions. For instance, do near-horizon symmetries [13] control the entropy counting in these models?

To summarize, we have shown that near-critical condensed matter systems can be considered as analogues of induced gravity theories and used to address a number of questions. The suggested approach avoids the main difficulty of gravity analogues [1, 2] introduced to study the Hawking radiation. This difficulty is the absence of the diffeomorphism invariance and the dynamical Einstein-like equations for the effective metric. In the new approach, the gravitational coupling can be defined as the entanglement entropy per unit area. Neither the effective metric nor its dynamics is needed. The coupling is defined from the response of the effective action to the conical singularity. It is a computable quantity which can be studied by analytical and numerical methods.

References

[1] Unruh W G 1981 *Phys. Rev. Lett.* **46** 1351

[2] Volovik G E 2003 *Universe in a Helium Droplet* (Oxford: Oxford University Press)
Barcelo C, Liberati S and Visser M 2005 Analogue Gravity *Preprint* gr-qc/0505065

[3] Vidal G, Latorre J I, Rico E and Kitaev A 2003 *Phys. Rev. Lett.* **90** 227902 (*Preprint* quant-ph/0211074)
Rico E 2005 *PhD Thesis* (*Preprint* quant-ph/0509037)

[4] Srednicki M 1993 *Phys. Rev. Lett.* **71** 666 (*Preprint* hep-th/9303048)

[5] Sakharov A D 1968 *Sov. Phys. Dokl.* **12** 1040

[6] Fursaev D V 1994 *Phys. Lett.* B **334** 53

[7] Fursaev D V and Solodukhin S N 1995 *Phys. Rev.* D **52** 2133

[8] Frolov V, Fursaev D and Zelnikov A 2003 *J. High Energy Phys.* JHEP 03(2003)038 (*Preprint* hep-th/0302207)

[9] Polyakov A M 1979 *Phys. Lett.* B **82** 247
Polyakov A M 1987 *Gauge Fields and Strings* (Chur: Harwood Academic)

[10] Latorre J I, Lütken C A, Rico E and Vidal G 2005 *Phys. Rev.* A **71** 034301 (*Preprint* quant-ph/0404120)

[11] Calabrese P and Cardy J 2004 *J. Stat. Mech.* JSTAT 06(2004)P002 (*Preprint* hep-th/0405152)

[12] Fursaev D V 2005 *Phys. Part. Nucl.* **36** 81–99 (*Preprint* gr-qc/0404038)

[13] Carlip S 2005 *Class. Quantum Grav.* **22** 3055 (*Preprint* gr-qc/0501033)